A New Approach to Van der Pol's Oscillator Problem

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In this paper, we will consider the Laplace decomposition method (LDM) for finding series solutions of nonlinear oscillator differential equations. The equations are Laplace transformed and the nonlinear terms are represented by He's polynomials. The solutions are compared with the numerical (fourth-order Runge–Kutta) solution and the solution obtained by the Adomian decomposition method. The suggested algorithm is more efficient and easier to handle as compared to the numerical method. The results illustrate that LDM is an appropriate method in solving the highly nonlinear equations.

Key words: Laplace Decomposition Method; Oscillator Differential Equation; He's Polynomials.

1. Introduction

Nonlinear ordinary differential equations arise in a wide variety of circumstances: a simple pendulum, oscillations in electrical circuits, oscillations of mechanical structures, molecular vibrations, the motion of particles in accelerators, planetary motion, and the effects of strong electromagnetic fields of atoms and molecules.

The Duffing–Van der Pol's equation provides an important mathematical model for dynamical systems having a single unstable fixed point, along with a single stable limit cycle. Examples of such phenomena arise in all of the natural and engineering sciences [1, 2] and in many physical problems [3, 4]. The problem of Van der Pol–Duffing oscillators have been studied extensively in various aspects, for example to the vibration amplitude control, synchronization dynamics, additive resonances, etc. [5 – 9]. The literature on the topic is quite extensive and hence can not be described here in detail. However, some most recent works of eminent researchers regarding the Van der Pol-Duffing oscillator may be mentioned in [10 – 15].

Most models of real-life problems, however, are still very difficult to solve. Therefore, approximate analytical solutions such as homotopy perturbation method [16 – 18], homotopy analysis method [19, 20], variational iteration method [21 – 23], differential transform method [24], Adomian decomposition method [25, 26], Laplace decomposition method [27, 28], and homotopy perturbation transform method [29] were introduced. In general, the solutions produced by the Laplace decomposition method (LDM) are as accurate as the solutions given by other methods like the control theory formalism [30]. It is well known that the control theory is an important mathematical theoretical tool, useful in a lot of applications in milling industry, economic models, robotics, electrical machine regulation, etc., which allows monitoring the solution of differential, difference, and hybrid systems in a prescribed way.

The Laplace transform is an elementary but useful technique for solving linear ordinary differential equations that is widely used by scientists and engineers for tackling linearized models. In fact, the Laplace transform is one of only a few methods that can be applied to linear systems with periodic or discontinuous driving inputs. Despite its great usefulness in solving linear problems, however, the Laplace transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. This paper considers the effectiveness of the Laplace decomposition algorithm in solving nonlinear oscillator differential equations.

2. The Laplace Decomposition Method

To illustrate the basic idea of this method, we consider the following general form of the second-order non-homogeneous nonlinear ordinary differential equation with initial conditions given by

$$f'' + b_1(x)f' + b_2(x)f = g(y), \quad f(0) = \alpha,\ f'(0) = \beta. \tag{1}$$

According to the Laplace decomposition method [27, 28], we apply the Laplace transform (denoted throughout this paper by L) on both sides of (1):

$$s^2 L[f] - s\alpha - \beta + L\left[b_1(x)f'\right] + L\left[b_2(x)f\right] = L[g(y)]. \tag{2}$$

Using the differentiation property of the Laplace transform, we have

$$L[f] = \frac{\alpha}{s} + \frac{\beta}{s^2} + \frac{1}{s^2}L[g(y)] - \frac{1}{s^2}L\left[b_1(x)f' + b_2(x)f\right]. \tag{3}$$

The Laplace decomposition method [27, 28] admits a solution in the form

$$f = \sum_{n=0}^{\infty} f_n, \tag{4}$$

so that the nonlinear term can be decomposed as

$$g(y) = Nf = \sum_{n=0}^{\infty} H_n, \tag{5}$$

for some He's polynomials H_n (see [31]) that are given by

$$H_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}p^n}\left[N\left(\sum_{i=0}^{\infty} p^i(f_i)\right)\right]_{p=0}, \quad n = 0,1,2,3,\ldots \tag{6}$$

Using (5) and (4) in (3), we get

$$L\left[\sum_{n=0}^{\infty} f_n\right] = \frac{\alpha}{s} + \frac{\beta}{s^2} + \frac{1}{s^2}L\left[\sum_{n=0}^{\infty} H_n\right] - \frac{1}{s^2}L\left[b_1(x)\sum_{n=0}^{\infty} f_n' + b_2(x)\sum_{n=0}^{\infty} f_n\right]. \tag{7}$$

Matching both sides of (7), we have the following relation:

$$L[f_0] = \frac{\alpha}{s} + \frac{\beta}{s^2}, \tag{8}$$

$$\begin{aligned} L[f_1] &= \frac{1}{s^2}L[H_0] - \frac{1}{s^2}L\left[b_1(x)f_0' + b_2(x)f_0\right], \\ L[f_2] &= \frac{1}{s^2}L[H_1] - \frac{1}{s^2}L\left[b_1(x)f_1' + b_2(x)f_1\right], \\ &\vdots \end{aligned} \tag{9}$$

In general the recursive relation for (9) is given by

$$L[f_{n+1}] = \frac{1}{s^2}L[H_n] - \frac{1}{s^2}L\left[b_1(x)f_n' + b_2(x)f_n\right], \quad n \geq 0. \tag{10}$$

Taking the inverse Laplace transform from both sides of (8)–(10), one obtains

$$f_0 = G(x) \tag{11}$$

and

$$f_{n+1}(x) = L^{-1}\left[\frac{1}{s^2}L[H_n] - \frac{1}{s^2}L\left[b_1(x)f_n' + b_2(x)f_n\right]\right], \quad n \geq 0, \tag{12}$$

where $G(x)$ represents the term arising from the source term and the prescribed initial condition.

3. Van der Pol's Oscillator Problem

Example 3.1 Consider the following Van der Pol's oscillator problem [12]:

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \frac{\mathrm{d}u}{\mathrm{d}t} + u + u^2\frac{\mathrm{d}u}{\mathrm{d}t} = 2\cos t - \cos^3 t \tag{13}$$

with the initial conditions

$$u(0) = 0,\ u'(0) = 1. \tag{14}$$

The exact solution of the above problem is given by

$$u(t) = \sin t. \tag{15}$$

We apply the Laplace transform to get

$$\begin{aligned} &s^2 u(s) - su(0) - u'(0) + su(s) - u(0) \\ &= L\left[2\cos t - \cos^3 t - u - u^2 \frac{\mathrm{d}u}{\mathrm{d}t}\right]. \end{aligned} \tag{16}$$

The initial condition now implies that

$$\begin{aligned} &s^2 u(s) - 1 + su(s) \\ &= L\left[2\cos t - \cos^3 t - u - u^2 \frac{\mathrm{d}u}{\mathrm{d}t}\right] \end{aligned} \tag{17}$$

and

$$\begin{aligned} u(s) = \frac{1}{s^2+s} + \frac{1}{s^2+s} L \\ \cdot \left[2\cos t - \cos^3 t - u - u^2 \frac{\mathrm{d}u}{\mathrm{d}t}\right] \end{aligned} \tag{18}$$

so that by applying the inverse Laplace transform, we have

$$\begin{aligned} u(t) = 1 - \mathrm{e}^{-t} + L^{-1}\left[\frac{1}{s^2+s} L \right. \\ \left. \cdot \left[2\cos t - \cos^3 t - u - u^2 \frac{\mathrm{d}u}{\mathrm{d}t}\right]\right]. \end{aligned} \tag{19}$$

Since a series solution of the form

$$u(t) = \sum_{n=0}^{\infty} u_n(t) \tag{20}$$

is assumed in the Laplace decomposition method, (20) is substituted into (19) to get

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(t) = 1 - \mathrm{e}^{-t} + L^{-1}\left[\frac{1}{s^2+s} L\left[2\cos t \right.\right. \\ \left.\left. - \cos^3 t - \sum_{n=0}^{\infty} u_n(t) - \sum_{n=0}^{\infty} H_{1n}(u)\right]\right], \end{aligned} \tag{21}$$

where $H_{1n}(u)$ are He's polynomials [31] that represent the nonlinear terms and satisfy

$$\sum_{n=0}^{\infty} H_{1n}(u) = u^2 u_t. \tag{22}$$

The first few components of He's polynomials, for example, are given by

$$\begin{aligned} &H_{10}(u) = u_0^2 u_{0t}, \\ &H_{11}(u) = 2u_0 u_1 u_{0t} + u_0^2 u_{1t}, \\ &H_{12}(u) = u_1^2 u_{0t} + 2u_0 u_2 u_{0t} + 2u_0 u_1 u_{1t} + u_0^2 u_{2t}, \\ &\vdots \\ &H_{1n}(u) = \sum_{i=0}^{n} \sum_{r=0}^{i} u_{n-it} u_r u_{i-r}. \end{aligned} \tag{23}$$

It is clear from (21) that the recursive relation is

$$u_0(t) = 1 - \mathrm{e}^{-t}, \tag{24}$$

$$\begin{aligned} &u_{n+1}(t) = L^{-1}\left[\frac{1}{s^2+s} L\left[2\cos t - \cos^3 t - u_n - H_{1n}\right]\right], \\ &n \geq 0, \end{aligned} \tag{25}$$

so that for $n = 0$, we have

$$u_1(t) = L^{-1}\left[\frac{1}{s^2+s} L\left[2\cos t - \cos^3 t - u_0 - H_{10}\right]\right],$$

$$\begin{aligned} u_1(t) = \frac{1}{120}\big(200 - 4\,\mathrm{e}^{-3t}(5 - 30\,\mathrm{e}^{t} + 57\,\mathrm{e}^{2t}) \\ -120t - 75\cos t + 3\cos 3t + 75\sin t - \sin 3t\big); \end{aligned} \tag{26}$$

therefore the solution $u(t)$ is given by

$$\begin{aligned} u(t) = \frac{1}{120}\big(320 - 4\,\mathrm{e}^{-3t}(5 - 30\,\mathrm{e}^{t} + 87\,\mathrm{e}^{2t}) \\ -120t - 75\cos t + 3\cos 3t + 75\sin t - \sin 3t\big) + \ldots \end{aligned} \tag{27}$$

Table 1 exhibits the errors obtained by applying the numerical fourth-order Runge–Kutta method, the Laplace decomposition method, and the Adomian decomposition method.

Example 3.2 Consider the nonlinear oscillator differential equation [12]

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - u + u^2 + \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 - 1 = 0$$

with the initial conditions

$$u(0) = 2, \; u'(0) = 0. \tag{28}$$

The exact solution of the above problem is given by

$$u(t) = 1 + \cos t. \tag{29}$$

Table 1. Comparison between numerical fourth-order Runge–Kutta method, Laplace decomposition method, and Adomian decomposition method for Example 3.1.

t	Numerical solution	LDM solution	ADM solution	Exact solution	Error = exact sol. − numer. sol.	Error = exact sol. − LDM sol.	Error = exact sol. − ADM sol.
0	0	0	0	0	0	0	0
0.1	0.0998	0.0998382	0.0998291	0.0998334	0.0000334	−0.0000048	0.00000043
0.2	0.1987	0.198754	0.198598	0.198669	−0.0000310	−0.0000850	0.00007100
0.3	0.2956	0.295971	0.295144	0.29552	−0.000080	−0.0004510	0.00037600
0.4	0.3896	0.390884	0.388161	0.389418	−0.0001820	−0.0014660	0.00125700
0.5	0.4796	0.483027	0.476167	0.479426	−0.0001740	−0.0036010	0.00325900
0.6	0.5648	0.572029	0.557442	0.564642	−0.0001580	−0.0073870	0.0072000
0.7	0.6444	0.657552	0.629971	0.644218	−0.0001820	−0.0133340	0.0142470
0.8	0.7174	0.739232	0.691389	0.717356	−0.0000440	−0.0218760	0.02596700
0.9	0.7832	0.816622	0.738922	0.783327	0.0001270	−0.0332950	0.04440500
1	0.841	0.889158	0.769345	0.841471	0.00047100	−0.0476870	0.07212600

Table 2. Comparison between numerical fourth-order Runge–Kutta method, Laplace decomposition method, and Adomian decomposition method for Example 3.2.

t	Numerical solution	LDM solution	ADM solution	Exact solution	Error = exact sol. − numer. sol.	Error = exact sol. − LDM sol.	Error = exact sol. − ADM sol.
0	2	2	2	2	0	0	0
0.1	1.995004	1.99498	1.99498	1.995	−0.00000400	0.0000250069	0.0000250069
0.2	1.980067	1.97967	1.97967	1.98007	−0.00000300	0.000400445	0.000400445
0.3	1.955336	1.95331	1.95331	1.95534	0.00000400	0.00203006	0.00203006
0.4	1.921061	1.91463	1.91463	1.92106	−0.00000100	0.00642846	0.00642846
0.5	1.877583	1.86185	1.86185	1.87758	−0.00000300	0.0157336	0.0157336
0.6	1.825336	1.79261	1.79261	1.82534	−0.00000820	0.0327244	0.0327244
0.7	1.764842	1.704	1.704	1.76484	−0.00000200	0.0608434	0.0608434
0.8	1.696707	1.59248	1.59248	1.69671	0.00000300	0.104225	0.104225
0.9	1.62161	1.45388	1.45388	1.62161	0.00000000	0.167726	0.167726
1	1.540302	1.28333	1.28333	1.5403	−0.00000200	0.256969	0.256969

In a similar way we have

$$u(s)=\frac{2}{s}+\frac{1}{s^3}+\frac{1}{s^2}L\left[u-u^2-\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2\right]. \quad (30)$$

The inverse of Laplace transform implies that

$$u(t)=2+\frac{t^2}{2}+L^{-1}\left[\frac{1}{s^2}L\left[u-u^2-\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2\right]\right]. \quad (31)$$

Proceeding as before, we obtain

$$u_{n+1}(t)=2+\frac{t^2}{2}+L^{-1}\left[\frac{1}{s^2}L[u_n-H_{2n}-H_{3n}]\right],\quad n\geq 0. \quad (32)$$

$H_{2n}(u)$ and $H_{3n}(u)$ are He's polynomials [31] that represent the nonlinear terms. Matching both sides of (32), the components of u can be obtained as follows:

$$u_0(t)=2+\frac{t^2}{2}, \quad (33)$$

$$u_1(t)=L^{-1}\left[\frac{1}{s^2}L[u_0-H_{20}-H_{30}]\right],\quad u_1(t)=-t^2-\frac{5t^4}{24}-\frac{t^6}{120}. \quad (34)$$

Therefore a series solution is obtained which reads

$$u(t)=2-\frac{t^2}{2}-\frac{5t^4}{24}-\frac{t^6}{120}-\ldots \quad (35)$$

Table 2 exhibits the errors obtained by applying the numerical fourth-order Runge–Kutta method, the Laplace decomposition method, and the Adomian decomposition method.

4. Conclusion

This paper uses He's polynomials to decompose the nonlinear terms in equations. A series solution for the nonlinear oscillator differential equations is derived by using the Laplace decomposition method (LDM). Such an analysis does not exist in the literature and the results obtained are new. These results are in good agreement with those given in [12] as well as those obtained by the numerical and the Adomian decomposition methods. This analysis therefore provides further support for the validity of LDM.

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